Exercise 8.5.2

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02407 Stochastic Processes

We model the velocity of a particle with an Ornstein-Uhlenbeck process $\{V_t\}_{t\geq 0}$ with parameters $\sigma^2 = 1$ and $\beta = 0.2$, and initial value $V_0 = 0$. Furthermore, we denote the position process of said particle as $\{S_t\}_{t\geq 0}$ and assume that the initial value is given by $S_0 = 0$. From eq. (8.64), we know the relationship between V_t and S_t is given as

$$S_t = S_0 + \int_0^t V_u du.$$

As stated on p. 444, under the conditions that $V_0 = S_0 = 0$, Theorem 8.4 yields that $S_t \sim \mathcal{N}(0, \sigma_t^2)$, where the variance is calculated as

$$\sigma_t^2 = \mathbb{V}[S_t] = \frac{\sigma^2}{\beta^2}\left[t - \frac{2}{\beta}\left(1 - e^{-\beta t}\right) + \frac{1}{2\beta}\left(1 - e^{-2\beta t}\right)\right].$$

We can then solve the problem easily by simple calculations:

$$\begin{aligned}\mathbb{P}\left(|S_t - S_0| > 1 | S_0 = 0, V_0 = 0\right) &= \mathbb{P}\left(|S_t| > 1 | S_0 = 0, V_0 = 0\right) \\ &= \mathbb{P}\left(S_t > 1 | S_0 = 0, V_0 = 0\right) + \mathbb{P}\left(S_t < -1 | S_0 = 0, V_0 = 0\right).\end{aligned}$$

You can proceed either by using that $\mathbb{P}\left(S_t > 1 | S_0 = 0, V_0 = 0\right) = 1 - \mathbb{P}\left(S_t \leq 1 | S_0 = 0, V_0 = 0\right)$ or note that $\mathbb{P}\left(S_t > 1 | S_0 = 0, V_0 = 0\right) = \mathbb{P}\left(S_t < -1 | S_0 = 0, V_0 = 0\right)$ due to the symmetry of the normal distribution around 0. We invoke the latter option. Hence,

$$\begin{aligned}\mathbb{P}\left(|S_t - S_0| > 1 | S_0 = 0, V_0 = 0\right) &= 2\mathbb{P}\left(S_t < -1 | S_0 = 0, V_0 = 0\right) \\ &= 2\mathbb{P}\left(\frac{S_t}{\sigma_t} < -\frac{1}{\sigma_t} | S_0 = 0, V_0 = 0\right) \\ &= 2\mathbb{P}\left(S_t^* < -\frac{1}{\sigma_t} | S_0 = 0, V_0 = 0\right),\end{aligned}$$

where the asterisk denotes a standardized variable. In conclusion,

$$\mathbb{P}\left(|S_t - S_0| > 1 | S_0 = 0, V_0 = 0\right) = 2\Phi\left(-\frac{1}{\sigma_t}\right).$$

Inserting the different time points, we obtain the values:

$$\begin{aligned}\sigma_1^2 = 0.2877, \quad &\mathbb{P}\left(|S_1 - S_0| > 1 | S_0 = 0, V_0 = 0\right) = 0.0311, \\ \sigma_{10}^2 = 95.1891, \quad &\mathbb{P}\left(|S_{10} - S_0| > 1 | S_0 = 0, V_0 = 0\right) = 0.4592, \\ \sigma_{100}^2 = 2312.5, \quad &\mathbb{P}\left(|S_{100} - S_0| > 1 | S_0 = 0, V_0 = 0\right) = 0.4917.\end{aligned}$$